A METHOD FOR THE ENGINEERING CALCULATION OF THE TEMPERATURE FIELDS OF SOLIDS UNDER CONDITIONS OF NATURAL HEAT EXCHANGE WITH THE ATMOSPHERE

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A method is developed for the engineering calculation of the temperature field of a solid [1] when its surface is subject to heat transfer by radiation, convection, and evaporation or condensation simultaneously. Solutions are given for an exponential initial temperature distribution and a two-layer system.

Under natural conditions heat transfer generally takes place on the surface of a solid by several mechanisms acting simultaneously (for example, convection, radiation, evaporation or condensation, etc.). This consideration, coupled with the fact that the role of each heat-transfer mechanism varies continuously with time by arbitrary laws, presents certain difficulties in calculating the temperature state of solids.

We now consider a method for the engineering calculation of the temperature field in a solid whose surface is subject to heat transfer by several mechanisms acting in concert, and for our initial data we specify the temperature–time dependence for air, solar radiation, and other parameters typical of the thermal regime on the surface of the given solid.

The heat flux across the surface of the solid can be expressed by means of the thermal balance equation:

$$-\lambda \frac{\partial T(0,\ \tau)}{\partial x} = q_s(\tau) - q_r(\tau) - q_c(\tau) - q_v(\tau). \tag{1}$$

Then, using the equations of heat- and mass-transfer theory, we can represent the quantities $q_s(\tau)$, $q_r(\tau)$, $q_c(\tau)$, and $q_v(\tau)$ each as an algebraic sum consisting of a linear function of the surface temperature and a time function given in the form of a power-law polynomial:

$$q_i(\tau) = k_i T(0,\ \tau) + \sum_{\mu=0}^{n} u_\mu^{(i)} \tau^\mu , \tag{2}$$

where k_i and u_μ are coefficients.

For instance, the convective heat flux may be determined from the relation

$$q_c(\tau) = \alpha(\tau)\left[T(0,\ \tau) - T_0(\tau)\right] . \tag{3}$$

In accordance with the condition set forth above we must approximate the function $T_0(\tau)$ on the investigated time interval by a power polynomial and average $\alpha(\tau)$, then use this average value in the calculations.

As another example, we determine the radiative heat transfer in the familiar way on the basis of the Stefan–Boltzmann law:

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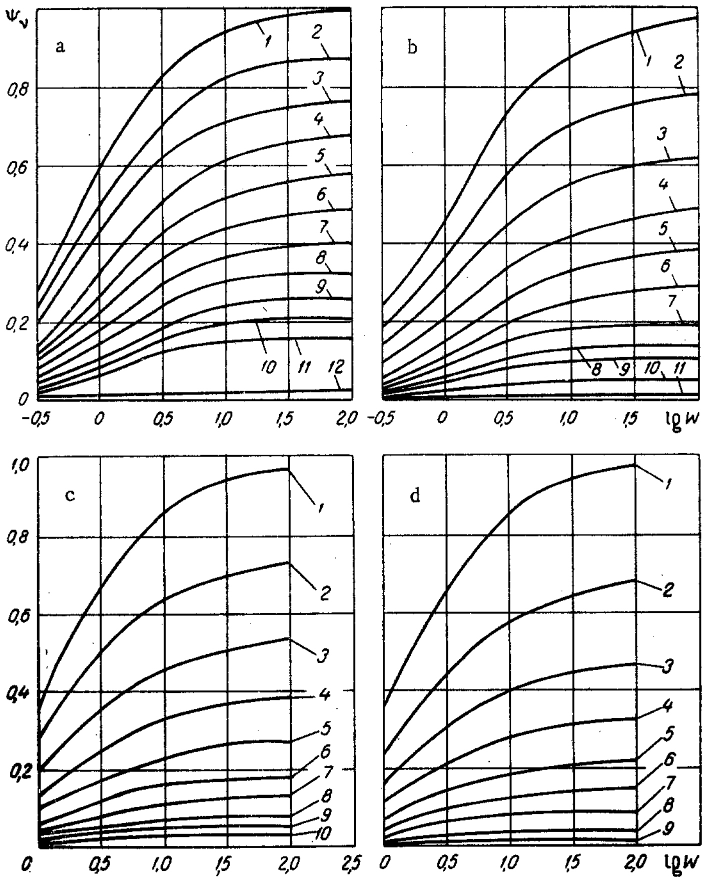

Fig. 1. Graphs of the functions Ψ_ν (W, Y) for v = 0 (a), 1 (b), 2 (c), and 3 (d). a: 1) Y = 0; 2) 0.1; 3) 0.2; 4) 0.3; 5) 0.4; 6) 0.5; 7) 0.6); 8) 0.7; 9) 0.8; 10) 0.9; 11) 1.0; 12) 1.5. b: 1) Y = 0; 2) 0.1; 3) 0.2; 4) 0.3; 5) 0.4; 6) 0.5; 7) 0.6; 8) 0.7; 9) 0.8; 10) 1.0; 11) 1.5. c: 1) Y = 0; 2) 0.1; 3) 0.2; 4) 0.3; 5) 0.4; 6) 0.5; 7) 0.6; 8) 0.7; 9) 0.8; 10) 1.0. d: 1) Y = 0; 2) 0.1; 3) 0.2; 4) 0.3; 5) 0.4; 6) 0.5; 7) 0.6; 8) 0.8; 9) 1.0.

$$q_r(\tau) = \varepsilon\sigma_0 \cdot 10^{-8}\left[T^4(0,\ \tau) - T^4_{\mathrm{bnd}}(\tau)\right]. \tag{4}$$

In radiative heat transfer under natural conditions the temperature $T(0, \tau)$ and $T_{\mathrm{bnd}}(\tau)$ vary between relatively narrow limits. This fact permits us to approximate $T^4(0, \tau)$ and $T^4_{\mathrm{bnd}}(\tau)$ accurately by linear functions and to write expression (4) in the form

$$q_r(\tau) = \varepsilon\sigma_0 n\left[T(0,\ \tau) - T_{\mathrm{bnd}}(\tau)\right]. \tag{5}$$

It has been shown in [1] that for the temperature interval 268 to 308°K the indicated transformation induces an error of at most 7%, where the coefficient n = 0.96.

Next, approximating the function $T_{\mathrm{bnd}}(\tau)$ on the given time interval by a power polynomial, we can easily reduce (5) to expression (2).

Thus, by reducing the variables $q_s(\tau)$, $q_r(\tau)$, $q_c(\tau)$, and $q_v(\tau)$ to expression (2) and then substituting them into the thermal balance equation (1) and consolidating coefficients of like powers τ^μ, we can determine the heat flux across the surface of the solid by the relation

$$-\lambda \frac{\partial T(0,\ \tau)}{\partial x} = \sum_{\nu=0}^{m} s_\nu \tau^\nu - NT(0,\ \tau). \tag{6}$$

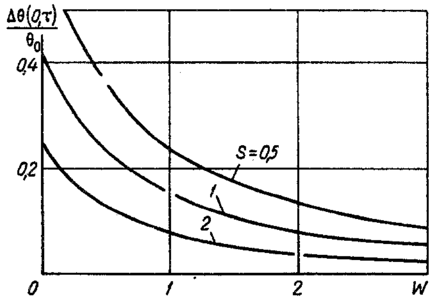

Fig. 2. Dimensionless variable $\Delta\theta(0,\tau)/\theta_0$ versus W for three values of S.

For m = 0 Eq. (6) is written

$$-\lambda \frac{\partial T(0, \tau)}{\partial x} = s_0 - NT(0, \tau). \tag{7}$$

Equation (7) corresponds to the special case in which the initial data are interpreted as parameters whose values are constant over the analytic time interval. It is important to mention that expression (7) has been derived in [2, 3].

We next examine several solutions of the heat-conduction equation under our boundary conditions (6) for some of the most important practical situations.

1. Semiinfinite Body Having a Uniform Initial Temperature

We denote the following:

$$\theta(x, \tau) = T(x, \tau) - T_\infty,$$

where

$$T_\infty = \text{const}.$$

The solution of the heat-conduction equation

$$\frac{\partial^2\theta(x, \tau)}{\partial x^2} - \frac{1}{a}\frac{\partial\theta(x, \tau)}{\partial\tau} = 0 \tag{8}$$

subject to the boundary conditions

$$\begin{gathered} -\frac{\partial\theta(0, \tau)}{\partial x} = \sum_{\nu=0}^{m} z_\nu \tau^\nu - H\theta(0, \tau), \\ \theta(\infty, \tau) = 0, \end{gathered} \tag{9}$$

in which

$$z_0 = \frac{s_0 - NT_\infty}{\lambda}, \quad z_\nu = \frac{s_\nu}{\lambda} \ (\nu = 1, 2, \ldots, m), \ H = \frac{N}{\lambda},$$

and the zero-valued initial condition

$$\theta(x, 0) = 0, \tag{10}$$

which ensues from the condition of a uniform initial temperature distribution [T(x, 0) = const], was found by an operational method and is written

$$\theta(x, \tau) = \frac{1}{H}\sum_{\nu=0}^{m} z_\nu \tau^\nu \Psi_\nu(W, Y), \tag{11}$$

where

$$\Psi_\nu = \frac{\nu!}{W^{2\nu}} \left\{ \sum_{\delta=0}^{2\nu} (-2W)^\delta i^\delta \operatorname{erfc} Y - \exp(W^2 + 2WY) \operatorname{erfc}(W+Y) \right\}, \tag{12}$$

$$W = H\sqrt{a\tau}, \quad Y = \frac{x}{2\sqrt{a\tau}}.$$

The functions Ψ_ν depend on the two dimensionless parameters W and Y. The values of these functions were calculated to facilitate their practical utilization. They are given in Fig. 1.

2. Semiinfinite Body Having an Exponential Initial Temperature Distribution

The solution of Eq. (8) subject to the boundary conditions (9) and the initial condition

$$\theta(x, 0) = \theta_0 \exp(-\Lambda x), \tag{13}$$

in which

$$\theta_0 = T(0, 0) - T_\infty,$$

was found by an operational method and is written

$$\theta(x, \tau) = \frac{1}{H} \sum_{\nu=0}^{m} z_\nu \tau^\nu \Psi_\nu(W, Y) + \theta_0 \exp(-Y^2)$$

$$\times \left[\frac{1}{2} \frac{S+W}{S-W} P\{u_1\} + \frac{1}{2} P\{u_2\} - \frac{W}{S-W} P\{u_3\} \right], \tag{14}$$

$$P\{u\} = \exp u^2 \operatorname{erfc} u,$$

$$u_1 = S + Y, \; u_2 = S - Y, \; u_3 = W + Y, \; S = \Lambda \sqrt{a\tau}.$$

The values of the function P{u} are given in [4].

Examining the solution (14), we notice that it consists of two parts; the first part [the first term, which is written as (11)] is determined by the boundary conditions on the surface, and the second part accounts for the influence of the initial distribution.

The role of the second part can be assumed by comparing (11) with (14) for x = 0:

$$\frac{\Delta\theta(0, \tau)}{\theta_0} = \frac{S}{S-W} \exp S^2 \operatorname{erfc} S - \frac{W}{S-W} \exp W^2 \operatorname{erfc} W. \tag{15}$$

The quantity $\Delta\theta(0, \tau)/\theta_0$ represents, in dimensionless form, the difference that arises in the calculation of the surface temperature of the body according to Eqs. (11) and (14). The behavior of this quantity as a function of W and S is illustrated by the graphs of Fig. 2, which were obtained from (15). It is apparent from Fig. 2 that for S > 1 and W > 1 we have

$$\frac{\Delta\theta(0, \tau)}{\theta_0} < 0.15.$$

Consequently, the influence of the nonuniformity of the initial distribution may be neglected in this case for the solution of practical problems, and the calculation can be carried out on the basis of (11).

3. Semiinfinite Body Having a Uniform Initial Temperature and on Its Surface a Plane Layer with Different Thermal Characteristics

This problem is formulated mathematically as follows. Solve the heat-conduction equations

$$\frac{\partial^2 \theta_1(x, \tau)}{\partial x^2} - \frac{1}{a_1} \frac{\partial \theta_1(x, \tau)}{\partial \tau} = 0, \quad -L \leqslant x \leqslant 0, \tag{16}$$

$$\frac{\partial^2\theta_2(x,\ \tau)}{\partial x^2} - \frac{1}{a_2}\ \frac{\partial\theta_2(x,\ \tau)}{\partial\tau} = 0, \quad x \geqslant 9 \tag{17}$$

subject to the initial condition

$$\theta_1(x,\ 0) = \theta_2(x,\ 0) = 0 \tag{18}$$

and the boundary conditions

$$-\frac{\partial\theta_1(-L,\ \tau)}{\partial x} = \sum_{\nu=0}^{m} z_\nu \tau^\nu - H\theta_1(-L,\ \tau), \tag{19}$$

$$\lambda_1 \frac{\partial\theta_1(0,\ \tau)}{\partial x} = \lambda_2 \frac{\partial\theta_2(0,\ \tau)}{\partial x}, \tag{20}$$

$$\theta_1(0,\ \tau) = \theta_2(0,\ \tau), \tag{21}$$

$$\theta_2(\infty,\ \tau) = 0. \tag{22}$$

The solutions of Eqs. (16) and (17) were also found by an operational method; they are written

$$\theta_1(x,\ \tau) = \frac{1}{H}\sum_{\nu=0}^{m} z_\nu \tau^\nu \left\{\Psi_\nu(W,\ Y_1) + \frac{b-1}{b+1}\Psi_\nu(W,\ Y_2)\right\}, \tag{23}$$

$$\theta_2(x,\ \tau) = \frac{2b}{H(b+1)}\sum_{\nu=0}^{m} z_\nu \tau^\nu \Psi_\nu(W,\ Y_3), \tag{24}$$

where

$$b = \frac{\lambda_2}{\lambda_1}\sqrt{\frac{a_1}{a_2}};$$

$$Y_1 = \frac{L+x}{2\sqrt{a_1\tau}},\ Y_2 = \frac{L-x}{2\sqrt{a_1\tau}},\ -L \leqslant x \leqslant 0,$$

$$Y_3 = \frac{L + x\sqrt{\dfrac{a_1}{a_2}}}{2\sqrt{a_1\tau}},\ x \geqslant 0.$$

Comparing (11), (14), (23), and (24), we note at once that the solutions are expressed in terms of the function $\Psi_\nu(W, Y)$. Therefore, for engineering calculations it is permissible to use the universal graphs of Fig. 1 in all three cases.

NOTATION

q_s	heat flux due to solar radiation;
q_r	resultant flux due to thermal radiation;
q_c	heat flux due to convection;
q_v	heat flux due to evaporation;
$T(x, \tau)$	temperature of a body at distance x at time τ;
σ_0	Stefan–Boltzmann constant;
ε	emissivity;
T_{bnd}	mean surface temperature of bodies with which a given body realizes radiative heat exchange;
α	convective heat-transfer coefficient;
T_0	air temperature;
λ	thermal conductivity;
a	thermal diffusivity.

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